

# A New Method to Study the Nonlinear Reaction-Diffusion Process in the Electroactive Polymer Film using Hyperbolic Function Method

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A mathematical model developed by Lyons and co-workers (Analyst, 121, (1996) 715–731) describes a substrate to form a complex with the immobilized catalyst is discussed. The hyperbolic function method is applied to solve the nonlinear equations in the electroactive polymer film. The resulting analytical expression of the substrate concentration is compared to the numerical results and previously available results. A satisfactory agreement is noted. The innovative method yields a compact set of analytical approximations that are easy to compute and simple to validate.

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**Keywords:** Hyperbolic function method, Mathematical modeling, Non-linear differential equation, Electroactive polymer film.

## 1. INTRODUCTION

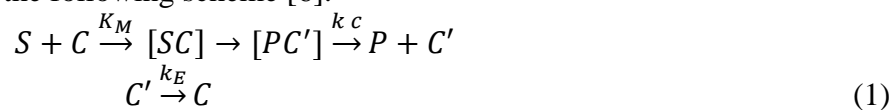
In the disciplines of electrocatalysis, chemical sensor technology, and associated energy storage technologies, electrode surfaces covered with electroactive polymer films are being used more and more frequently. Over the past 20 years, several simplified mathematical models that describe electrocatalysis in electroactive polymer films have been developed [1,2]. The study frequently involves formulating and solving nonlinear reaction-diffusion equations, which leads to the development of approximative analytical methods for the current amperometric response.

This investigation is not simple because one deals with modelling nonlinear reaction/diffusion processes in thin films. Several authors have presented insightful analyses of recent advancements in this field, including Hillman [3], Lyons [4-6], Evans [7], Wring and Hart [8], Andrieux and Saveant [9], Albery [10-13], Bartlett [14,15], Rajendran and collaborators [16-27]. For polymer-modified electrodes, more challenging and complex models have been put forth by Albery and Hillman [10] and Andrieux and Saveant [9]. Theoretically, [10,11] the latter processes are not well defined. A thorough application of the experimental procedures was provided by Bartlett et al. [14].

Theoretical models of the steady-state current of a polymer-modified electrode, which exhibits a rate law defined by the Michaelis-Menten equation, have been produced by Albery et al. [10] and Lyons and coworkers [1]. Using the Akbar Ganji Method [29-33], Dharmalingam et al. [28] could determine the analytical expression of the concentration of the substrate. So, for steady-state conditions, we examined a bounded diffusion problem. In this work, we propose a simple and efficient approach to solve the steady-state nonlinear differential equation that arises in the context of the electroactive polymer film.

## 2. THE MATHEMATICAL FORMULATION FOR THE PROBLEM.

We just provide a quick explanation of the model employed here because the model's specifics have been comprehensively discussed in [6]. We take into consideration a thin, uniformly thick electrocatalytically active polymer film that has been placed on a support electrode. The active sites are expected to be evenly distributed throughout the polymer matrix. We further presume that the layer is electrically conductive and that charge percolation through the film does not affect the rate. We further presume that the layer is electrically conductive and that charge percolation through the film does not affect the rate. Additionally, we assume that the catalyst C and substrate S will react with Michaelis-Menten kinetics under the following scheme [6].



The catalytically active form of the immobilized catalyst is represented by C and C'. According to [SC], the enzyme-substrate complex, and [PC], the product-enzyme complex, S and P stand for substrate and product, respectively. The Michaelis-Menten constant is denoted by the letters  $K_M$ , while the catalytic constant is denoted by  $k_c$  [6].

$D_S$  and  $D'_S$ , stand for a substrate's diffusion coefficient in a film and the Nernst diffusion layer. The partition coefficient of and the partitioning of the substrate into the polymer film is assumed to be true. The reaction kinetics is Michaelis-Menten form. The governing reaction/diffusion equation is expressed using the non-dimensional parameters as follows [6]:

$$\frac{d^2u(\chi)}{d\chi^2} + \frac{n}{x} \frac{du(\chi)}{d\chi} - \frac{\gamma u(\chi)}{1+\alpha u(\chi)} = 0 \quad (2)$$

where,

$$u = \frac{S}{\kappa S^\infty}; \quad \chi = \frac{x}{L}; \quad \alpha = \frac{\kappa S^\infty}{K_M}; \quad \gamma = \Phi^2 = \frac{k L^2}{D_S} \quad (3)$$

where  $u$  denotes dimensionless substrate concentration, and  $\chi = \frac{x}{L}$  species the distance parameters.  $\nabla^2$  represent the Laplacian operator. The first two terms accounts for substrate diffusion inside the polymer matrix, and the third term is non-linear in concentration which describes Michaelis-Menten kinetics. Additionally, a defines the relationship between the substrate concentration in the film  $\kappa s^\infty$  and the Michaelis constant  $K_M$ , which allows us to measure the degree of unsaturation or saturation of the catalytic kinetics. In Eq. (3), the last parameter specifies the relationship between the pseudo-first-order constant  $k$  for the chemical reaction occurring at the active site and the rate of constant' diffusion of the substrate through the polymer matrix. The constant  $n = 0$  for planar,  $n = 1$  for Cylindrical and  $n = 2$ , for Spherical Geometry. As a result, it provides a method for determining how substrate dispersion affects reaction kinetics. The following is an explanation of the beginning and boundary conditions:

$$\chi = 0; \frac{du(\chi)}{d\chi} = 0 \quad (4)$$

$$\chi = 1; u(\chi) = 1 \quad (5)$$

The Taylor series method [34,35] is applied to solve the nonlinear reaction/diffusion equation for the steady-state to obtain the expression for the concentration profile of the substrate through the polymer layer. We will demonstrate how to quickly, easily, and effectively construct solutions to reaction/diffusion differential equations using the Taylor series technique. Now that the concentration profile has been integrated, the current reaction can be assessed as follows:

$$y = \frac{iL}{nFAD_S K_M} = \alpha \left( \frac{du}{d\chi} \right)_{\chi=1} \quad (6)$$

By solving the non-linear equation using the Taylors series method, recently Usharani et al. [22] obtained substrate concentration as follows.

$$u(\chi) \approx u(0) + \frac{\gamma u(0)}{(n+1)(1+\alpha u(0))} \frac{\chi^2}{2!} + \frac{3\gamma^2 u(0)}{(n+1)(n+3)(1+\alpha u(0))^3} \frac{\chi^4}{4!} + \frac{15\gamma^3 u(0)[(n+1)-2\alpha u(0)(n+3)]}{(n+1)^2(n+3)(n+5)(1+\alpha u(0))^5} \frac{\chi^6}{6!} \quad (7)$$

where  $u(0)$  can be obtained by solving the below equation (8):

$$1 = u(0) + \frac{\gamma u(0)}{(n+1)(1+\alpha u(0))} \frac{1^2}{2!} + \frac{3\gamma^2 u(0)}{(n+1)(n+3)(1+\alpha u(0))^3} \frac{1^4}{4!} + \frac{15\gamma^3 u(0)[(n+1)-2\alpha u(0)(n+3)]}{(n+1)^2(n+3)(n+5)(1+\alpha u(0))^5} \frac{1^6}{6!} \quad (8)$$

### 3. THE CONCENTRATION OF SUBSTRATES USING THE HYPERBOLIC FUNCTION METHOD

Several semi-analytical techniques, including Taylor's series method [46–50], the Adomian decomposition method [36], the homotopy analysis method [37,38], the homotopy perturbation method [39–41], the variational iteration method [42], the generalized differential transformation method [43], a residual method [44], the hyperbolic function method [45], some other new analytical method [51], and the Akbari–Ganji method (AGM) have been proposed the analytical or computational solutions for nonlinear differential equations. He [52] recently investigated many mathematical techniques for the problem of fractal derivatives to the fourth-order nonlinear integral boundary value problems. The nonlinear differential equations that control this system are analytically solved in this work using the hyperbolic method. Additionally, this approach makes it simple to resolve complex nonlinear equations without the use of challenging mathematical techniques. The

simple analytical expression for the steady-state substrate concentration is obtained by solving the nonlinear Eqs. 2 and 8 using this method (Appendix A) as follows:

$$u(\chi) = u(0) \cosh \left( \cosh^{-1} \left( \frac{1}{u(0)} \right) \cdot \chi \right) \tag{9}$$

where  $u(0)$  can be obtained from the equation (12)

$$1 = u(0) + \frac{\gamma u(0)}{(n+1)(1+\alpha u(0))} \frac{1^2}{2!} + \frac{3\gamma^2 u(0)}{(n+1)(n+3)(1+\alpha u(0))^3} \frac{1^4}{4!} + \frac{15\gamma^3 u(0)[(n+1)-2\alpha u(0)(n+3)]}{(n+1)^2(n+3)(n+5)(1+\alpha u(0))^5} \frac{1^6}{6!} \tag{10}$$

The Eqn. (9) is the new closed- form of analytical expressions of a concentration of the substrate  $u(\chi)$  for all values of the saturation  $\alpha$  and diffusion parameter  $\gamma$ , respectively. The dimensionless form of the normalized current  $y$  is

$$y = \alpha \left( \frac{du}{d\chi} \right)_{\chi=1} = \alpha \left[ \cosh^{-1} \left( \frac{1}{u(0)} \right) (1 + u(0)) \sqrt{\frac{(1-u(0))}{(1+u(0))}} \right] \tag{11}$$

$$= \alpha \sqrt{1 - (u(0))^2} \left[ \cosh^{-1} \left( \frac{1}{u(0)} \right) \right] \tag{12}$$

The above result is valid when  $u(0) < 1$ . This result is also obtained via the method of Akbari-Ganji (Appendix-B).

#### 4. DISCUSSION

The new approximate analytical expressions for a substrate in electroactive polymer films are given in equation (9) for all values of the parameters  $\alpha \left( = \frac{\kappa s^\infty}{K_M} \right)$  and  $\gamma \left( = \phi^2 = \frac{\kappa L^2}{D_S} \right)$ . The layer thickness  $L$ , substrate diffusion coefficient  $D_S$  within the polymer film, and reaction/diffusion parameter  $\gamma$ , which governs the relationship between the speed of a chemical reaction within the layer and the rate of substrate diffusion across the film, are the intriguing parameters. The parameter  $\alpha$  quantifies the degree of unsaturation/saturation of the catalytic kinetics. The analytical expression of concentration (9) and current (12) for planar and other electrodes are summarized in Table 1.

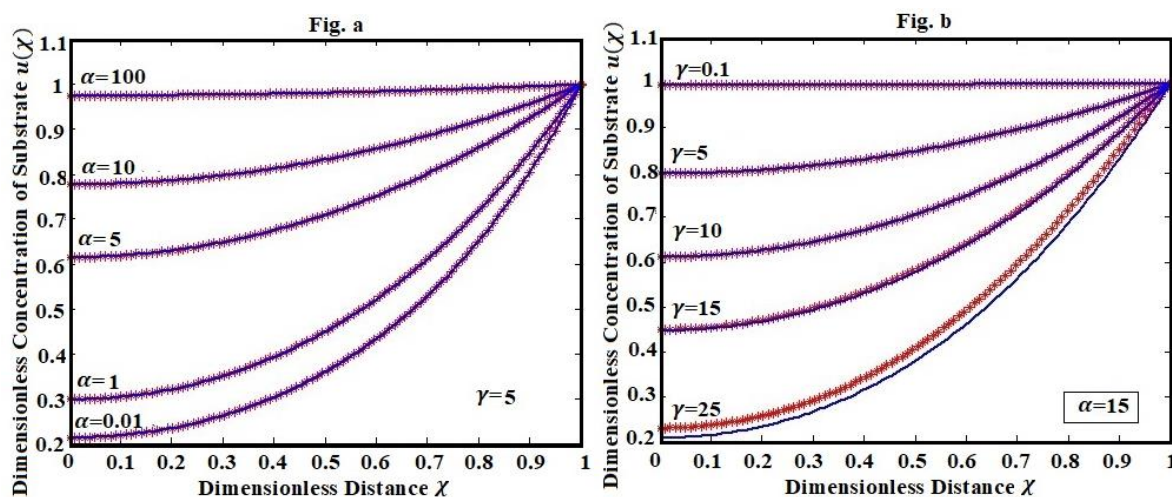
**Table 1.** The numerous analytical equations for the planar electrode's concentration and current

| S. No | Ref.               | Concentration $u$ and steady-state current $y$  | Value of $\gamma$ and $\alpha$   |
|-------|--------------------|---|----------------------------------|
| 1     | Albery et al. [11] | $u(\chi) = \cosh \left( \gamma^{\frac{1}{2}} \chi \right) - \sinh \left( \gamma^{\frac{1}{2}} \chi \right) \tanh \left( \gamma^{\frac{1}{2}} \right)$<br>(13) | $\alpha \leq 1$                  |
|       |                    | $y(\alpha, \gamma) = \alpha [\sqrt{\gamma}] \tanh[\sqrt{\gamma}]$<br>(14)   |                                  |
|       |                    | $u(\chi) = \left[ \frac{\gamma}{2\alpha} \right] \chi^2 - \frac{\gamma}{\alpha} \chi + 1$<br>(15)   | $\alpha \leq 1$                  |
|       |                    | $y(\alpha, \gamma) = (2\gamma\alpha)^{1/2}$<br>(16)   |                                  |
|       |                    | $y(\alpha, \gamma) = \left( (2\gamma [\alpha - 1n(1 + \right)$  | All values $\gamma$ and $\alpha$ |

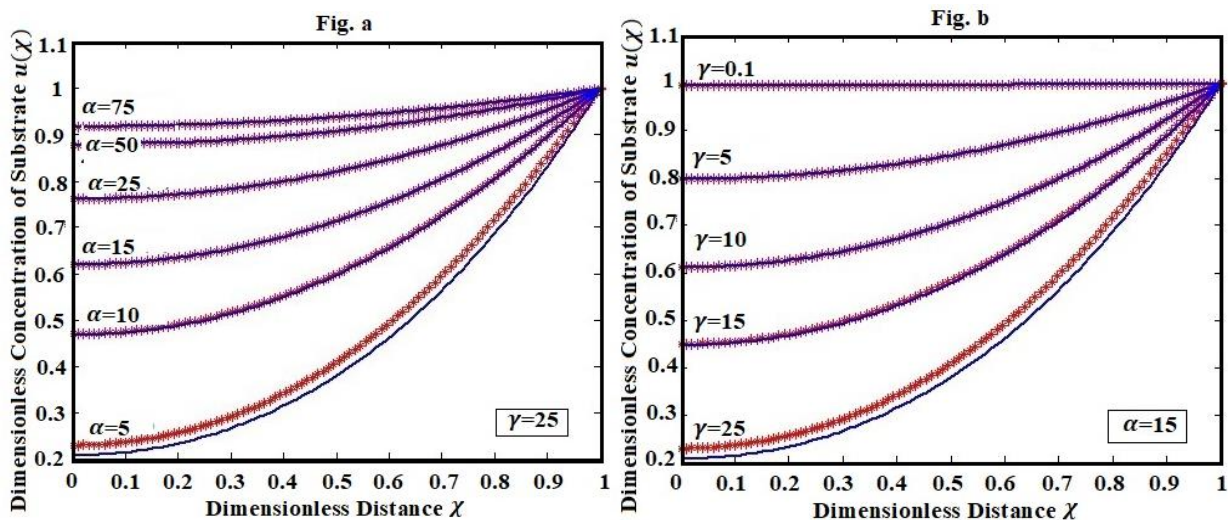
|  |  |  |                                     |
|--|--|--|-------------------------------------|
|  |  | $\alpha)]^{\frac{1}{2}}) \tanh\left(\frac{\gamma^{\frac{1}{2}}\alpha}{(1+\alpha)(2[\alpha-1n(1+\alpha)])^{\frac{1}{2}}}\right) \quad (17)$   |                                     |
| 2  | Lyons et al. [5]                           | $u(\chi) = \frac{\cosh(\sqrt{\gamma} \chi) \operatorname{sech}(\sqrt{\gamma})}{(18)}$  |                                     |
|  |  | $y(\alpha, \gamma) = \alpha\sqrt{\gamma} \tanh \sqrt{\gamma} \quad (19)$   | $\alpha \leq 1$                     |
|  |  | $u(\chi) = 1 + \frac{\delta}{2\alpha} (\chi^2 - 1) \quad (20)$   | $\alpha > 1$                        |
|  |  | $y(\alpha, \gamma) = \gamma - \frac{\gamma^2}{3\alpha} \quad (21)$   |                                     |
| 3  | Rahamathunissa and Rajendran [16]          | $u(\chi) = 1 + \frac{4\gamma}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left\{ \cos\left[\left(\frac{n+1}{2}\right) \pi \chi\right] \right\}}{(2n+1) \left[ \left(\frac{n+1}{2}\right)^2 \pi^2 + \gamma \right]} \quad (22)$  | $\alpha \leq 1$                     |
|  |  | $y(\alpha, \gamma) = \alpha\delta - \frac{4\alpha\gamma^2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1) \left[ \left(\frac{n+1}{2}\right)^2 \pi^2 + \gamma \right]} \quad (23)$  |                                     |
|  |  | $= \alpha\sqrt{\gamma} \tanh(\sqrt{\gamma})$   |                                     |
|  |  | $u(\chi) = 1 + \frac{\delta}{2\alpha} (\chi^2 - 1) \quad (24)$   | $\alpha > 1$                        |
|  |  | $y(\alpha, \gamma) = \gamma - \frac{\gamma^2}{3\alpha} \quad (25)$   |                                     |
|  |  | $y_{ss}(\alpha, \gamma) = \frac{\alpha\sqrt{\gamma} \tanh(\sqrt{\gamma}) + \alpha^2 \frac{(\tanh(\sqrt{\gamma}))^2}{1 - \frac{1}{3}\sqrt{\gamma} \tanh(\sqrt{\gamma})}}{1 + \frac{\alpha\sqrt{\gamma} \tanh(\sqrt{\gamma})}{\left(1 - \frac{1}{3}\sqrt{\gamma} \tanh(\sqrt{\gamma})\right)\sqrt{\gamma}} + \frac{\alpha^2 \sqrt{\gamma} (\tanh(\sqrt{\gamma}))^2}{\left(1 - \frac{1}{3}\sqrt{\gamma} \tanh(\sqrt{\gamma})\right)\gamma}} \quad (26)$ | $\alpha > 1$                        |
|  |  | $y_{ss}(\alpha, \gamma) = \frac{\alpha\sqrt{\gamma} \tanh(\sqrt{\gamma})}{1 + \sqrt{\frac{\alpha}{2}} \tanh(\sqrt{\gamma})} \quad (27)$  | $\alpha \leq 1$                     |
| $y_{ss}(\gamma) = \frac{\gamma}{1 + \sqrt{\frac{\gamma}{2\alpha}}} \quad (28)$ | $\alpha > 1$<br>for all values of $\gamma$ |  |                                     |
| 4  | Dharmalingam et al. [28]                   | $u(\chi) = \frac{\cosh\left(\sqrt{\frac{\gamma}{1+\alpha}} \chi\right)}{\cosh\left(\sqrt{\frac{\gamma}{1+\alpha}}\right)} \quad (29)$  | All values of $\gamma$ and $\alpha$ |
|  |  | $y(\alpha, \gamma) = \alpha \frac{\sinh\left(\sqrt{\frac{\gamma}{1+\alpha}}\right) \sqrt{\frac{\gamma}{1+\alpha}}}{\cosh\left(\sqrt{\frac{\gamma}{1+\alpha}}\right)} \quad (30)$   | All values of $\gamma$ and $\alpha$ |
| 5  | Usha Rani et al. [22]                      | $u(\chi) = u(0) + \frac{\gamma u(0)\chi^2}{(1+\alpha u(0))2!} + \frac{\gamma^2 u(0)\chi^4}{(1+\alpha u(0))^3 4!} + \frac{\gamma^3 u(0)(1-6\alpha u(0))\chi^6}{(1+\alpha u(0))^5 6!} + \dots$   | All values of $\gamma$ and $\alpha$ |

|   |                  |  |                        |
|---|------------------|--|------------------------|
|   |                  | $\frac{\gamma^4 u(0) (96 \alpha^2 u(0)^2 - 20 \alpha u(0) - 16 \alpha u(0) - 6 \alpha^2 u(0)^2 + 1) \chi^8}{(1 + \alpha u(0))^{78!}} + \dots$ <p>(31)</p>  |                        |
|   |                  | $y = \alpha \gamma \left[ u(0) + \frac{\gamma u(0)}{(n+1)(1+\alpha u(0))} \frac{1}{3!} + \frac{3\gamma^2 u(0)}{(n+1)(n+3)(1+\alpha u(0))^3} \frac{1}{5!} + \frac{15\gamma^3 u(0)[(n+1)-2\alpha u(0)(n+3)]}{(n+1)^2(n+3)(n+5)(1+\alpha u(0))^5} \frac{1}{7!} + \dots \right]$ <p>(32)</p> | $\alpha \leq 1$        |
|   |                  | $y = \gamma \left[ u(0) + \frac{\gamma u(0)}{(n+1)(1+\alpha u(0))} \frac{1}{3!} + \frac{3\gamma^2 u(0)}{(n+1)(n+3)(1+\alpha u(0))^3} \frac{1}{5!} + \frac{15\gamma^3 u(0)[(n+1)-2\alpha u(0)(n+3)]}{(n+1)^2(n+3)(n+5)(1+\alpha u(0))^5} \frac{1}{7!} + \dots \right]$ <p>(33)</p>        | $\alpha > 1$           |
| 6 | This work. (HFM) | $u(\chi) = u(0) \cosh \left( \cosh^{-1} \left( \frac{1}{u(0)} \right) \cdot \chi \right)$ <p>(34)</p>  | All values of $\alpha$ |
|   |                  | $y = \alpha \sqrt{1 - (u(0))^2} \left[ \cosh^{-1} \left( \frac{1}{u(0)} \right) \right]$ <p>(35)</p>   | All values of $\alpha$ |

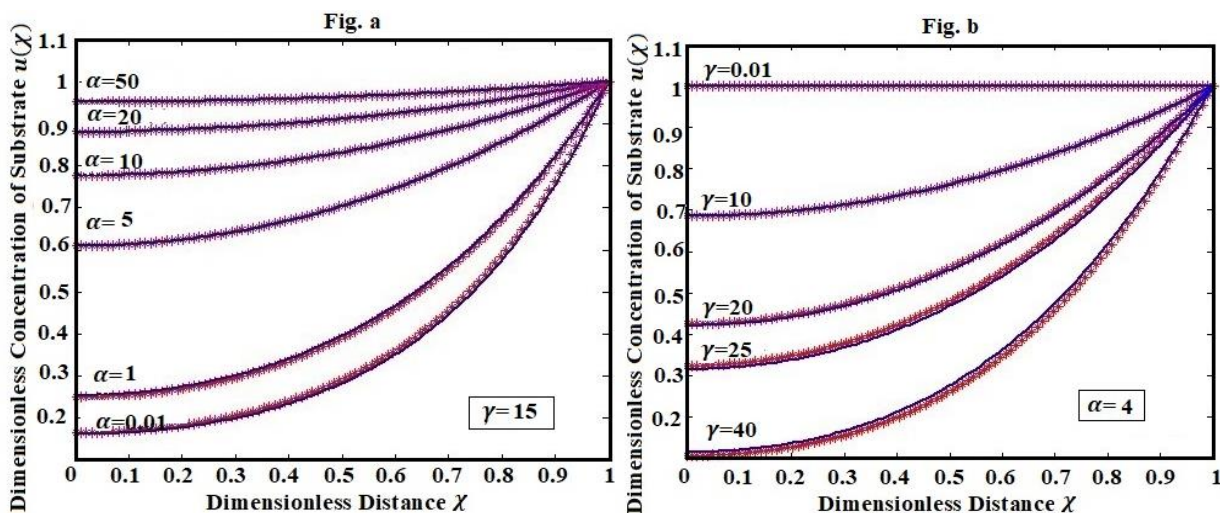
In Figs. 1-3 for the planar, spherical, and cylindrical electrodes, the numerical results for different values of the parameters are compared with our analytical expression of the substrate concentration  $u(\chi)$ . The graphs lead to the conclusion that as it  $\alpha$  rises or  $\gamma$  decreases, the substrate concentration on the electrode surface ( $\chi=0$ ) increases.



**Figure 1.** Comparison of analytical result (Eq. (9)) with numerical results of substrate concentration  $u(\chi)$  for planar electrode ( $n = 0$ ) for the different values parameters. (a) various values of  $\alpha$  and fixed values of  $\gamma$ . (b) various values of  $\gamma$  and fixed values of  $\alpha$ . The analytical results are indicated by (\*\*\*\*\*), and the numerical results are indicated by ( ).



**Figure 2.** Comparison of analytical result (Eq. (9)) with numerical results of substrate concentration  $u(\chi)$  for cylindrical electrode ( $n = 1$ ) for the different values parameters. (a) various values of  $\alpha$  and fixed values of  $\gamma$ . (b) various values of  $\gamma$  and fixed values of  $\alpha$ . The analytical results are indicated by (\*\*\*\*\*), and the numerical results are indicated by (●●●●●).



**Figure 3.** Comparison of analytical result (Eq. (9)) with numerical results of substrate concentration  $u(\chi)$  for spherical electrode ( $n = 2$ ) for the different values parameters. (a) various values of  $\alpha$  and fixed values of  $\gamma$ . (b) various values of  $\gamma$  and fixed values of  $\alpha$ . The analytical results are indicated by (\*\*\*\*\*), and the numerical results are indicated by (●●●●●).

**Table 2.** Comparison of normalized steady-state concentrations (Eqns. (8) and (29)) and simulation results for various values of parameters.

| $\chi$         | $\alpha = 0.1, \gamma = 5$ and $u(0) = 0.221092$ |                 |                |                  |                | $\alpha = 0.1, \gamma = 25$ and $u(0) = 0.014744$ |                 |                |                  |                |
|----------------|--|-----------------|----------------|------------------|----------------|---|-----------------|----------------|------------------|----------------|
|                | Numerical Value                                  | HFM of Eqn. (8) | Error % of HFM | AGM of Eqn. (29) | Error % of AGM | Numerical Value                                   | HFM of Eqn. (8) | Error % of HFM | AGM of Eqn. (29) | Error % of AGM |
| 0.0            | 0.2193   | 0.2211          | 0.82           | 0.2339           | 6.66           | 0.0139  | 0.0147          | 5.76           | 0.0170           | 22.30          |
| 0.1            | 0.2248   | 0.2265          | 0.76           | 0.2392           | 6.41           | 0.0157  | 0.0166          | 5.73           | 0.0190           | 21.02          |
| 0.2            | 0.2416   | 0.2431          | 0.62           | 0.2555           | 5.75           | 0.0216  | 0.0227          | 5.09           | 0.0253           | 17.13          |
| 0.3            | 0.2704   | 0.2715          | 0.41           | 0.2834           | 4.81           | 0.0332  | 0.0346          | 4.22           | 0.0376           | 13.25          |
| 0.4            | 0.3127   | 0.3133          | 0.19           | 0.3242           | 3.68           | 0.0533  | 0.0553          | 3.75           | 0.0585           | 9.76           |
| 0.5            | 0.3706   | 0.3702          | 0.11           | 0.3799           | 2.51           | 0.0873  | 0.0900          | 3.09           | 0.0930           | 6.53           |
| 0.6            | 0.4467   | 0.4450          | 0.38           | 0.4528           | 1.37           | 0.1438  | 0.1470          | 2.23           | 0.1490           | 3.62           |
| 0.7            | 0.5448   | 0.5409          | 0.72           | 0.5465           | 0.31           | 0.2372  | 0.2397          | 1.05           | 0.2395           | 0.97           |
| 0.8            | 0.6693   | 0.6621          | 1.08           | 0.6650           | 0.64           | 0.3908  | 0.3891          | 0.44           | 0.3856           | 1.33           |
| 0.9            | 0.8259   | 0.8133          | 1.53           | 0.8140           | 1.44           | 0.6422  | 0.6267          | 2.41           | 0.6209           | 3.32           |
| 1.0            | 1.0000   | 1.0000          | 0.00           | 1.0000           | 0.00           | 1.0000  | 1.0000          | 0.00           | 1.0000           | 0.00           |
| <b>Average</b> |  |                 | <b>0.60</b>    |                  | <b>3.05</b>    | <b>Average</b>                                    |                 |                | <b>3.07</b>      | <b>9.02</b>    |

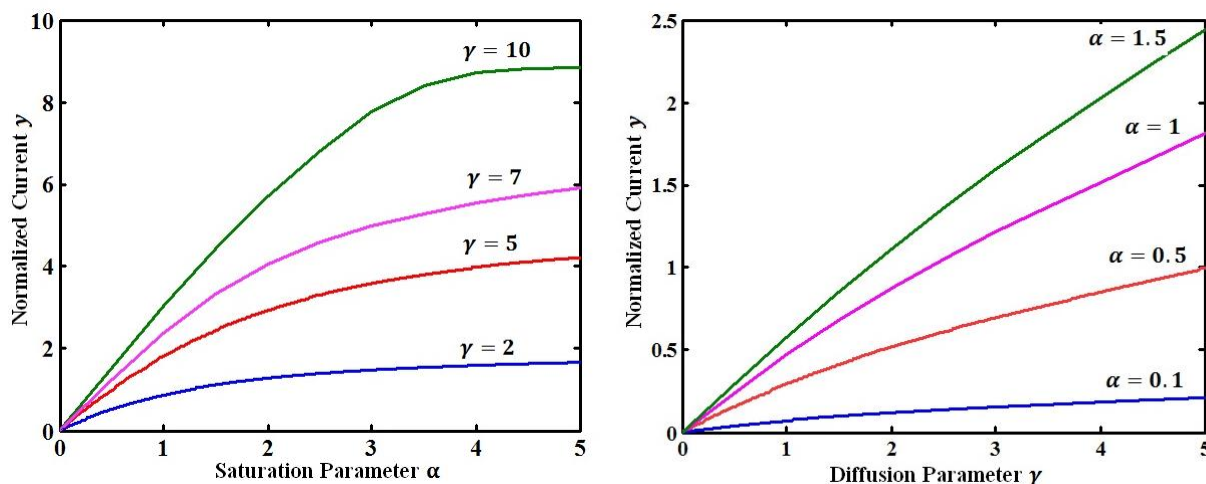
**Table 3.** Comparison of normalized steady-state concentrations (Eqns. (8) and (29)) and simulation results for various values of parameters.

| $\chi$         | $\alpha = 0.1, \gamma = 0.1$ and $u(0) = 0.956065$ |                 |                |                  |                | $\alpha = 25, \gamma = 5$ and $u(0) = 0.904165$ . |                 |                |                  |                |
|----------------|--|-----------------|----------------|------------------|----------------|---|-----------------|----------------|------------------|----------------|
|                | Numerical Value                                    | HFM of Eqn. (8) | Error % of HFM | AGM of Eqn. (29) | Error % of AGM | Numerical Value                                   | HFM of Eqn. (8) | Error % of HFM | AGM of Eqn. (29) | Error % of AGM |
| 0.0            | 0.9561   | 0.9561          | 0.00           | 0.9562           | 0.01           | 0.9042  | 0.9042          | 0.00           | 0.9110           | 0.75           |
| 0.1            | 0.9565   | 0.9565          | 0.00           | 0.9566           | 0.01           | 0.9051  | 0.9051          | 0.00           | 0.9117           | 0.73           |
| 0.2            | 0.9578   | 0.9578          | 0.00           | 0.9580           | 0.02           | 0.9081  | 0.9081          | 0.00           | 0.9145           | 0.70           |
| 0.3            | 0.9601   | 0.9601          | 0.00           | 0.9602           | 0.01           | 0.9130  | 0.9128          | 0.02           | 0.9189           | 0.65           |
| 0.4            | 0.9632   | 0.9632          | 0.00           | 0.9633           | 0.01           | 0.9198  | 0.9195          | 0.03           | 0.9250           | 0.57           |
| 0.5            | 0.9672   | 0.9672          | 0.00           | 0.9673           | 0.01           | 0.9286  | 0.9281          | 0.05           | 0.9330           | 0.47           |
| 0.6            | 0.9721   | 0.9721          | 0.00           | 0.9722           | 0.0            | 0.9393  | 0.9387          | 0.06           | 0.9427           | 0.36           |
| 0.7            | 0.978  | 0.9780          | 0.00           | 0.9780           | 0.00           | 0.9521  | 0.9511          | 0.11           | 0.9543           | 0.23           |
| 0.8            | 0.9847   | 0.9847          | 0.00           | 0.9847           | 0.00           | 0.9667  | 0.9655          | 0.12           | 0.9676           | 0.09           |
| 0.9            | 0.9923   | 0.9923          | 0.00           | 0.9924           | 0.01           | 0.9834  | 0.9818          | 0.16           | 0.9829           | 0.05           |
| 1.0            | 1.0000   | 1.0000          | 0.00           | 1.0000           | 0.00           | 1.0000  | 1.0000          | 0.00           | 1.0000           | 0.00           |
| <b>Average</b> |  |                 | <b>0.00</b>    |                  | <b>0.01</b>    | <b>Average</b>                                    |                 |                | <b>0.05</b>      | <b>0.42</b>    |

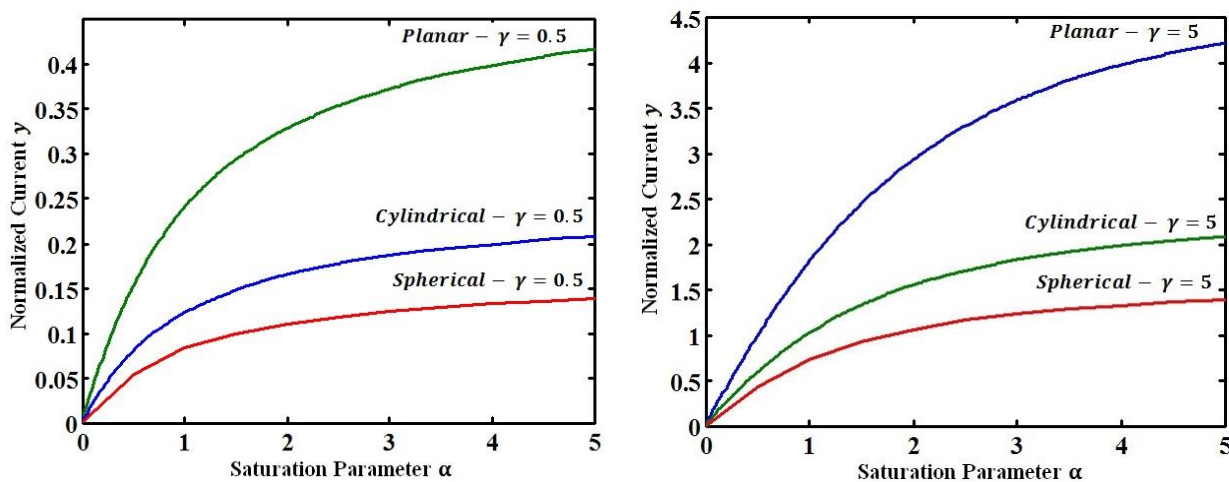
As a result of electrocatalysis at polymer-modified electrodes with various geometries(Planar, cylindrical, and spherical), a unique steady-state current response is produced, as shown by Eq (12). The error percentage of our results using the hyperbolic function method is lower than that of any other results. It can also be deduced from Tables 4-5, that the current increases as the saturation parameter and diffusion parameters rise. In contrast, the maximum average error between the simulation result and our analytical result (Eq. (8)) is 3%. The simulation result and the prior analytical result have a maximum average error of 9%. This mathematical model can also be applied to measure the current flow between electrodes when a redox reaction occurs and a biosensor with multi enzymes system with



non-Michaelis-Menten kinetics. The reaction-diffusion mechanism in biofuel cells and bioreactors can also be employed using this mathematical model for amperometric sensors.



**Figure 3.** Plot of dimensionless normalized current  $y$  versus dimensionless (a) saturation parameter  $\alpha$  (b) diffusion parameter  $\gamma$  for various values of other parameter.



**Figure 4.** Plot of dimensionless normalized current  $y$  versus dimensionless (a) saturation parameter  $\alpha$  (b) diffusion parameter  $\gamma$  for various values of other parameter.

**Table 4.** Comparison of steady-state current  $y$  with the different analytical and numerical results at various values of the parameters.

| $\alpha$ | Numerical Value | $\gamma = 0.5$  |                |                                       |                |  |                | $\gamma = 1$    |   |                |                                       |                |  |                |
|----------|-----------------|---|----------------|---------------------------------------|----------------|--|----------------|-----------------|---|----------------|---------------------------------------|----------------|--|----------------|
|          |                 | Lyons et al. Eq. (19) For $\alpha \leq 1$ (21) For $\alpha > 1$ | % of deviation | HFM of Eqn. (35) For $\forall \alpha$ | % of deviation | Dharmalingam et al. eqn. (30) For $\forall \alpha$ | % of deviation | Numerical Value | Lyons et al. Eq. (19) For $\alpha \leq 1$ (21) For $\alpha > 1$ | % of deviation | HFM of Eqn. (35) For $\forall \alpha$ | % of deviation | Dharmalingam et al. eqn. (30) For $\forall \alpha$ | % of deviation |
| 0.01     | 0.00433         | 0.00431   | 0.46404        | 0.00427                               | 1.40515        | 0.00427  | 1.40515        | 0.00747         | 0.00762   | 1.9685         | 0.00753                               | 0.79681        | 0.00756  | 1.19048        |
| 0.05     | 0.02049         | 0.02153   | 4.83047        | 0.02075                               | 1.25302        | 0.02063  | 0.67863        | 0.03694         | 0.03808   | 2.9937         | 0.03684                               | 0.27144        | 0.03666  | 0.76378        |
| 0.1      | 0.04069         | 0.04305   | 5.48199        | 0.04011                               | 1.44603        | 0.03963  | 2.67474        | 0.07105         | 0.07616   | 6.7096         | 0.07177                               | 1.0032         | 0.07068  | 0.52349        |
| 5        | 0.41103         | 0.48333   | 14.9587        | 0.41716                               | 1.46946        | 0.40546  | 1.37375        | 0.82443         | 0.93333   | 11.668         | 0.83539                               | 1.31196        | 0.78993  | 4.36748        |

|                |         |             |         |         |             |         |             |                |         |             |         |             |         |             |
|----------------|---------|-------------|---------|---------|-------------|---------|-------------|----------------|---------|-------------|---------|-------------|---------|-------------|
| 10             | 0.46033 | 0.49166     | 6.37229 | 0.45549 | 1.06259     | 0.44778 | 2.80272     | 0.90136        | 0.96667 | 6.7562      | 0.91295 | 1.26951     | 0.88251 | 2.13595     |
| 500            | 0.50000 | 0.49983     | 0.03402 | 0.49904 | 0.19237     | 0.49884 | 0.23254     | 1.00002        | 0.99667 | 0.3362      | 0.99082 | 0.92852     | 0.98684 | 1.33558     |
| 1000           | 0.50000 | 0.49992     | 0.01601 | 0.49952 | 0.09609     | 0.49942 | 0.11614     | 1.00001        | 0.99966 | 0.0351      | 1.00008 | 0.007       | 0.99867 | 0.13418     |
| 2000           | 0.48000 | 0.49996     | 3.99232 | 0.49976 | 3.95389     | 0.49971 | 3.94428     | 1.00001        | 0.99984 | 0.017       | 1.00004 | 0.003       | 0.99934 | 0.06704     |
| <b>Average</b> |         | <b>4.52</b> |         |         | <b>1.36</b> |         | <b>1.65</b> | <b>Average</b> |         | <b>3.81</b> |         | <b>0.69</b> |         | <b>1.32</b> |

**Table 5.** Comparison of steady-state current  $y$  for various values of  $\alpha$  when  $\gamma=5$ .

| $\alpha$       | Numerical Value | Lyons et al. Eq. (19) For $\alpha \leq 1$ (21) For $\alpha > 1$ | % of deviation | HFM of Eqn. (35) For $\forall \alpha$ | % of deviation | Dharmalingam et al. eqn. (30) For $\forall \alpha$ | % of deviation |
|----------------|-----------------|---|----------------|---------------------------------------|----------------|--|----------------|
| 0.01           | 0.02157         | 0.02185   | 1.28146        | 0.02158                               | 0.04634        | 0.02174  | 0.78196        |
| 0.1            | 0.21008         | 0.21856   | 3.87994        | 0.21296                               | 1.35237        | 0.20728  | 1.35083        |
| 5.5            | 3.99702         | 3.48485   | 14.6971        | 4.29918                               | 7.02832        | 3.40052  | 17.5414        |
| 10.5           | 4.45128         | 4.20635   | 5.82286        | 4.67316                               | 4.74796        | 4.00144  | 11.2419        |
| 100.5          | 5.02542         | 4.91708   | 2.20334        | 4.97044                               | 1.10614        | 4.87102  | 3.16977        |
| 500.5          | 5.00502         | 4.98335   | 0.43485        | 4.99415                               | 0.21765        | 4.97352  | 0.63335        |
| 1000.5         | 5.00251         | 4.99167   | 0.21716        | 4.99708                               | 0.10866        | 4.98671  | 0.31684        |
| 2000.5         | 4.80121         | 4.99584   | 3.89584        | 4.99854                               | 3.94775        | 4.99334  | 3.84773        |
| <b>Average</b> |                 |   | <b>4.73</b>    |                                       | <b>2.64</b>    |  | <b>6.03</b>    |

## 5. CONCLUSIONS

An amperometric sensor's mathematical model is described. Through Michaelis-Menten reaction kinetics, the sensing components in a surface-deposited polymer film make contact with the substrate. Analytical solutions have been found for a nonlinear time-independent partial differential equation. With the hyperbolic function approach, analytical expressions were reported for the substrate's concentration and the current's steady-state response. When compared to other analytical techniques, this method is straightforward, has a simple solution, and yields accurate results. This method can quickly solve other nonlinear boundary value problems in the physical and chemical sciences.

### CREDIT AUTHORSHIP CONTRIBUTION STATEMENT

S. Rekha: Methodology, Resources, Visualization, Supervision, Validation, Writing - original draft. R. Usha Rani: Investigation and Project administration, Software. L. Rajendran: Conceptualization, Writing - review & editing, Investigation, Project administration, Data curation, M. E. Lyons: Software, Formal analysis, Investigation, Supervision.

### DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**Appendix A:** Analytical solution of nonlinear equations (Eqs.2 and 3) using the hyperbolic function method (HFM).

The approximate analytical solution of equation (2) is considered as follows:

$$u(\chi) = A \cosh(m\chi) + B \sinh(m\chi) \tag{A1}$$

The derivative of Eq. (A1) is

$$u'(\chi) = A m \sinh(m\chi) + B m \cosh(m\chi) \tag{A2}$$

The boundary conditions are,

$$u'(0) = 0 \tag{A3}$$

$$u(1) = 1 \tag{A4}$$

Now, we can assume that one more boundary condition is  $u(0)$ . (A5)

Using the boundary condition (A5),

$$A = u(0) \tag{A6}$$

Using the boundary condition (A4),

$$1 = u(0) \cosh m + B \sinh m \tag{A7}$$

Using the boundary condition (A3), we get

$$B = 0 \tag{A8}$$

Now, substituting  $B = 0$  in Eq. (A7) becomes,

$$1 = u(0) \cosh m \tag{A9}$$

Now substituting  $A = u(0)$  and  $B = 0$  in Eq. (A1), we get

$$u(\chi) = u(0) \cosh(m\chi) \tag{A10}$$

Using Eq. (A9)

$$m = \cosh^{-1} \left( \frac{1}{u(0)} \right) \tag{A11}$$

∴ The solution to the equation is

$$u(\chi) = u(0) \cosh \left( \cosh^{-1} \left( \frac{1}{u(0)} \right) \cdot \chi \right) \tag{A12}$$

**Appendix B. Relation to the result obtained via the method of Akbari-Ganji.**

Using the Akbari-Ganji method it is readily shown that the expression for the concentration profile of reactant is:

$$u = \frac{\cosh(m\chi)}{\cosh(m)} \tag{B1}$$

where m is given by the following expression:

$$m \cong \sqrt{\frac{\gamma}{(1+n)(1+\alpha)}} \tag{B2}$$

Furthermore

$$u(\chi) = u(0) \cosh[m\chi] \tag{B3}$$

From eq.(A9) we can show that

$$u(0) = \frac{1}{\cosh m} = \operatorname{sech} m \tag{B4}$$

Clearly we note that

$$u(\chi) = \operatorname{sech} m \cosh(m\chi) \tag{B5}$$

Clearly, eq.(B4) is equivalent to eq.(A12) since  $\cosh^{-1}\left(\frac{1}{u(0)}\chi\right) = \cosh^{-1}(\cosh m\chi) = m\chi$  and

$\cosh\left(\cosh^{-1}\left(\frac{1}{u(0)}\right)\chi\right) = \cosh(m\chi)$  as required. Furthermore it has been shown that the reaction flux is given by

$$y = \alpha \left( \frac{du}{d\chi} \right)_1 = u(0)m \sinh m = m \tanh m \quad (\text{B6})$$

We recall the following identity:

$$\tanh m = \sqrt{1 - \operatorname{sech}^2 m} \quad (\text{B7})$$

Hence the normalized current is given by:

$$y = \alpha m \sqrt{1 - \operatorname{sech}^2 m} = \alpha m \sqrt{1 - u^2(0)} \quad (\text{B8})$$

Recalling that  $\cosh m = \frac{1}{u(0)}$  then we note that,

$$m = \cosh^{-1}\left(\frac{1}{u(0)}\right) \quad (\text{B9})$$

If eq.(B9) is substituted into eq.(B8) we immediately obtain eq.(12) in the text. The advantage of the Akbari-Ganji result is that  $m$  may be immediately related to the reaction/diffusion parameter  $\gamma$  and the saturation parameter  $\alpha$  and to the geometry of the problem via the  $n$  value, where  $n = 0, 1, 2$  for planar, cylindrical and spherical diffusion respectively.

## Reference

1. M.E.G. Lyons. *Electroactive Polymer Electrochemistry: Fundamentals, Part I*, Plenum Press, Springer Publications, New York, (1994)237.
2. M.E.G. Lyons. *Electroactive Polymer Electrochemistry: Fundamentals, Part I*, Plenum Press, Springer Publications, New York, (1994)1.
3. A.R. Hillman, in: R.G. Linford. A.R. Hillman, in: R.G. Linford (Ed.), *Electrochemical Science and Technology of Polymers*, Elsevier Applied Sciences, Amsterdam, (1987)103.
4. M.E.G. Lyons, T. Bannon, G. Hinds, S. Rebouillat. *The Analyst*, 123(1998)1947.
5. M.E.G. Lyons, J.C. Greer, C.A. Fitzgerald, T. Bannon, P.N. Barlett. *The Analyst*, 121(1996)715.
6. M.E.G. Lyons, A. Michas, P.N. Bartlett. *The Analyst*, 117(8)(1992)1271.
7. G.P. Evans, in: H. Gerisher, C.W. Tobias. *Advances in Electrochemical Science and Engineering*, V.C.H, Weinheim, 1(1990)1.
8. S.A. Wring, J.P. Hart. *The Analyst*, 117(8)(1992)1215.
9. P. Andrieux, J.M. Saveant, R.W. Murray. *Techniques of Chemistry Series, Molecular Design of Electrode Surfaces*, Wiley-Interscience, New York, vol. XXII(1992)207.
10. W.J. Albery, A.R. Hillman. *Roy. Soc. Chem. Annu. Rep. C*, 78(1981)377.
11. W.J. Albery, A.E.G. Cass, Z.X. Shu. *Biosens. Bioelectron*, 5 (1990) 367.
12. W.J. Albery, A.E.G. Cass, Z.X. Shu. *Biosens. Bioelectron*, 5(1990)379.
13. W.J. Albery, A.E.G. Cass, Z.X. Shu. *Biosens. Bioelectron*, 5(1990)397.
14. P.N. Bartlett, R.G. Whotaker. *J. Electroanal. Chem. Interfacial Electrochem.*, 224(1987)27.
15. W.J. Albery, P.N. Barlett, D.H. Craston. *J. Electroanal. Chem. Interfacial Electrochem.*, 194(1985)223.
16. G. Rahamathunissa, L. Rajendran. *J. Theor. Comput. Chem.*, 7(1)(2008)113.

17. L. Rajendran. *J. Theor. Comput. Chem.*, 5 (1)(2006)11.
18. L. Rajendran. *Electrochim. Acta*, 51(2006)5407.
19. A. Meena, L. Rajendran. *J. Electroanal. Chem.*, 650(2010)143.
20. A. Eswari, L. Rajendran. *J. Electroanal. Chem.*, 641(2010)35.
21. A. Meena, L. Rajendran. *J. Electroanal. Chem.*, 647(2010)103.
22. R. Usha Rani and L. Rajendran. *Chem. Phys. Lett.* 754(2020)137573.
23. R. Usha Rani and L. Rajendran, M.E.G. Lyons. *J. Electroanal. Chem.*, 886(2021)115103.
24. Lilly Clarence Mary, Ramu Usha Rani, Athimoolam Meena and Lakshmanan Rajendran. *Int. J. Electrochem. Sci.*, 16(2021)151037.
25. S.A. Salai Sivasundari, R. Usha Rani, M.E.G. Lyons and L. Rajendran. *Int. J. Electrochem. Sci.*, 17(2022)2.
26. K. Lakshmi Narayanan, R. Shanthi, R. Usha Rani, M.E.G. Lyons and L. Rajendran. *Int. J. Electrochem. Sci.*, 17(2022)2.
27. Marwan Abukhaled, R. Usha Rani and L. Rajendran. *Math. Model. Eng. Probl*, 8(4)(2021)527.
28. K.M. Dharmalingam, M. Veeramuni. *J. Electroanal. Chem.*, 844(2019)1.
29. M.R. Akbari, D.D. Ganji, A.K. Rostami, M. Nimafar. *J. Mar. Sci. Appl*, 14(2015)30.
30. N.B. Meresht, D.D. Ganji. *Int. J. Appl. Comput. Math*, 3(2017)1507.
31. S. Berkan, S.R. Hoseini, D.D. Ganji. *Propul. Power Res*, 6(2017)277.
32. M. Sheikholeslami, P. Jalili, D.D. Alex. *Eng. J*, 57(2018)587.
33. H. Mirgolbabaee, S.T. Ledari, N.M. Zadeh, D.D. Ganji. *J. Taibah Univ. Sci.*, 11(2017)1110.
34. J.H. He, F.Y. Ji. *J. Math. Chem.*, 57(2019)1932.
35. C.H. He, Y. Shen, F.Y. Ji and J.H. He. *Fractals*, 28(2019)01.
36. G. Adomian. *J. Math. Anal. Appl*, 135(2)(1988)501.
37. Shijun Liao. *Appl. Math. Comput*, 147(2)(2004)499.
38. Shijun Liao. *Appl. Math. Comput*, 169(2)(2005)1186.
39. Ji-Huan He. *Comput. Methods Appl. Mech. Engrg*, 178(1999)257.
40. Ji-Huan He. *Int. J. Nonlinear Mech*, 35(1)(2000)37.
41. R. Saravanakumar, P. Pirabaharan, Marwan Abukhaled, L. Rajendran. *J. Phys. Chem.B*, 124(2020)443.
42. Ji-Huan He. *Appl. Math. Comput*. 114(2000)115.
43. Z. Odibat, S. Momani. *Appl. Math. Lett*, 21(2008)194.
44. K. Saranya, V. Mohan, L. Rajendran. *J. Math. Chem.*, 58(2020)1230.
45. K. Nirmala, B. Manimegalai, L. Rajendran. *Int. J. Electrochem. Sci.*, 15(2020)5682.
46. E. Miletics, G.Molnárka. *Int. J. Comp. Meth-sing*, 4(2004)105.
47. P. Rentrop. *Numer. Math*, 31(1978)359.
48. E. Miletics, G. Molnárka. *HEJ: Appl Numer Math*, (2003)1.
49. Svetlin G. Georgiev, İnci M. Erhan. *Appl. Math. Comput*, 378(2020)125200.
50. Ghiocel Groza, Mohsen Razzaghi. *Comput. Math. Appl*, 66(7)(2013)1329.
51. M Chitra Devi, P Pirabaharan, Marwan Abukhaled, L Rajendran. *Electrochim. Acta*, 345(2020)136175.
52. He JH. *Int. J. Numer. Method. H*, 30 (11)(2020)4933.
53. M. R. Akbari, D. D. Ganji, M. Nimafar, A. R. Ahmadi. *Front. Mech. Eng*, 9(2014)390.
54. A.K. Rostami, M.R. Akbari, D.D. Ganji, S. Heydari. *Cent. Eur. J. Eng*, 4(2014)357.
55. M.R. Akbari, D.D. Ganji, A. Majidian, A.R. Ahmadi. *Front. Mech. Eng*, 9(2014)177.
56. M.R. Akbari, M. Nimafar, D.D. Ganji, H. Karimi Chalmiani. *Appl. Math. Comput*, 298(2017)95.
57. Navid Balazadeh Meresht, D.D. Ganji. *Int. J. Appl. Comput. Math*, 3(2017)1507.