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Mathematical Modelling of Unsteady Flow of Gas in a Semi-Infinite Porous Medium

M. Lilly Clarance Mary[1], *R. Saravanakumar*[4], *D. Lakshmanaraj*[2,*], *L. Rajendran*[3,*], *.M.E.G. Lyons*[5]

[1] Department of Mathematics, Fatima College (Autonomous), Madurai, India.
[2] Department of Mathematics, Sethu Institute of Technology, Kariapatti, India
[3] Department of Mathematics, AMET (Deemed to be University), Kanathur, Chennai, India
[4] Department of Mathematics, Vidhyaa Giri College of Arts and Science, Puduvayal, Karaikudi, India
[5]School of Chemistry & AMBER National Centre, University of Dublin, Trinity College Dublin, Dublin 2, Ireland.
[*]E-mail: lakshmanaraj@sethu.ac.in, raj_sms@rediffmail.com

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The research aims to find a way to solve the nonlinear problem in unsteady isothermal gas flow through a semi-infinite medium using a simple and effective method. This nonlinear equation is solved using a novel analytical technique called new approach of homotopy perturbation (NHPM) and Akbari-Ganji methods (AGM) to obtain the analytical expression of unsteady gas flow of the liquid through a porous medium. Comparing our results with other numerical and analytical methods validates our method's efficiency and accuracy. This method is very effective and concise, and this simple and closed-form of theoretical expression contain only one or two terms.

Keywords: Mathematical modeling, Nonlinear equation, New approach of homotopy perturbation method. Akbari-Ganji method, Unsteady gas flow.

1. INTRODUCTION

Fluid (liquid or gas) flow through porous media frequently occurs in various engineering applications, including electrochemical systems, composites manufacturing, oil production, geothermal engineering, nuclear thermal disposal, soil pollution, and so on. Electrochemical reactors, batteries, supercapacitors, fuel cells, and other electrochemical devices are widely used in modern society. The electrochemical systems generally involve a common and fundamental process in which the electrolyte fluid flows through the porous electrode. Newman has done outstanding work on the porous electrode model during the last few decades [1-3].

Unsteady gas flow in a porous medium resembles unsteady heat conduction in solids [4–10] . The gas flow modelling through porous media is quite valuable because of its importance in investigating the gas-solid processes. Although porous media have many applications in various areas of applied sciences and engineering, the detailed mathematical analysis to obtain an accurate analytical solution describing gas flow through a porous medium is lacking.

On the other hand, some researchers [6,11] examined the unsteady flow of gas through a porous material using numerical methods. A quasi-uniform grid and standard finite difference algorithms have been used by Fazio et al. [11] to find an accurate solution for unsteady gas flow in a semi-infinite porous medium.

A summary of previously available analytical and numerical techniques suitable for the analysis of this problem is provided in Table-1. However, all the presented analytical results were only moderately accurate. Furthermore, the calculations become increasingly more difficult as the order of terms increases, particularly beyond the second-order term [12-30]. As a consequence, this communication is focused on establishing a simple and closed-form analytical expression for unsteady gas flow through a porous medium which will be of general validity. These analytical results are useful to understand and optimize of the reaction diffusion processes in unsteady gas flow through a porous material.

2. MATHEMATICAL FORMULATION OF THE UNSTEADY FLOW OF GAS THROUGH A POROUS MEDIUM

The first study was conducted on the unsteady gas flow through a semi-infinite porous medium when the gas is initially injected at a uniform pressure of P_0. The pressure at the outflow face is abruptly reduced from P_0 to P_1 for the case of diffusion into a vacuum and, after that, maintained at this lower pressure. We focused on the flow of a liquid sample of wastewater obtained from a standard/conventional dyeing procedure to develop a numerical solution of the unsteady gas flow through a nanoporous material. An ideal porous pipe comprised of nanoparticles bound together and constituting a porous matrix has been proposed [28] based on the idea of fluid movement within a porous medium. To describe the flow of gas through a semi-infinite (x= 0 to x=∞) porous medium, a nonlinear partial differential equation has been formulated [31]:

$$\frac{\partial}{\partial x}\left(P\frac{\partial P}{\partial x}\right) = A\frac{\partial P}{\partial t} \tag{1}$$

The boundary conditions are represented by:

$$P(x,0) = P_0, \qquad 0 < x < \infty \tag{2}$$

$$P(0,t) = P_1, \qquad 0 \le t < \infty. \tag{3}$$

To acquire alike solution, Waltman [32] have introduced the new independent variable:

$$z = \frac{x}{\sqrt{t}}\left(\frac{A}{4P_0}\right)^{1/2}, \tag{4}$$

whereas the dimension-free dependent variable u is given by:

$$u(z) = A^{-1}\left(1 - \frac{P^2(z)}{P_0^2}\right), \tag{5}$$

where A is the real parameter defined as:

$$A = 1 - \frac{P_1^2}{P_0^2}. \tag{6}$$

According to previous work [31], the actual parameter A should exhibit values in the range (0;1).

This problem can be expressed in terms of the following non-linear differential equation :

$$\frac{d^2u(z)}{dz^2} + \frac{2z}{\sqrt{1-Au(z)}}\frac{du(z)}{dz} = 0 \quad (7)$$

The following boundary conditions must be obeyed :

$u = 1$ at $z = 0$ (8)

$u = 0$ at $z = \infty$ (9)

The Homotopy Preturbation Method is now applied to derive a closed form analytical solution to the boundary value problem presented in eq.(7)-(9).

3. APPLYING THE NHPM METHOD FOR SOLVING UNSTEADY GAS PROBLEM

Nonlinear differential equations are useful in physics, chemistry, biology, and economics, among other subjects. Because the analytical methods used to solve these equations are constrained and can only be applied in specific situations, they can't be used to solve equations in a wide range of realistic situations.

He [33-35] proposed the homotopy perturbation method in 1998. This technique has been applied to a variety of linear and nonlinear problems. This method is used to solve nonlinear boundary value problems [36-38].In physical and chemical sciences, this author used the homotopy perturbation approach to solve nonlinear equations [39-45]

The new homotopy perturbation method is used to give the approximate solutions of the non-linear eq.(7). To find the solution of eq.(7),we construct the homotopy as follows:

$$(1-p)\left[\frac{d^2u}{dz^2} + \frac{2z}{\sqrt{1-Au(z=0)}}\frac{du}{dz}\right] + p\left[\frac{d^2u}{dz^2} + \frac{2z}{\sqrt{1-Au(z)}}\frac{du}{dz}\right] = 0 \quad (10)$$

The analytical solution of eq.(10)is

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (11)$$

Substituting eqs.(11) into eq.(10) we get

$$(1-p)\left[\frac{d^2}{dz^2}(u_0 + pu_1 + p^2u_2 + \dots) + \frac{2z}{\sqrt{1-A}}\frac{d}{dz}(u_0 + pu_1 + p^2u_2 + \dots)\right]$$

$$+p\left[\frac{d^2}{dz^2}(u_0 + pu_1 + p^2u_2 + \dots) + \frac{2z\frac{d}{dz}(u_0+pu_1+p^2u_2+\dots)}{\sqrt{1-A(u_0+pu_1+p^2u_2+\dots)}}\right] = 0 \quad (12)$$

Comparing the coefficients of like powers of p in eq.(12)we get

$$p^0: \frac{d^2u_0}{dz^2} + \frac{2z}{\sqrt{1-A}}\frac{du_0}{dz} = 0 \quad (13)$$

The initial and boundary approximations is as follows

$$u_0(0) = 1, u_0(\infty) = 0 \quad (14)$$

Solving the eq. (13) and using the boundary conditions eq. (14), we obtain the following results for the first iteration:

$$u_0(z) = 1 - erf\left(\frac{z}{(1-A)^{1/4}}\right) = erfc\left(\frac{z}{(1-A)^{1/4}}\right)$$

Therefore, we have

$$u(z) \approx u_0(z) = erfc\left(\frac{z}{(1-A)^{1/4}}\right) \quad (16)$$

A higher number of iterations makes $u(z)$ converge to the exact solution.

4. APPLYING THE AKBARI -GANJI METHOD (AGM) FOR SOLVING UNSTEADY GAS PROBLEM

Based on the mass balance equations (11-16), Akbari-Ganji's method [46,47] is used to solve the boundary value problem. This method has a minimum number of unknowns as compared to other methods.It is an appropriate and simple method to the nonlinear differential equations [48-51]. It is a particular case of exponential function method which is proposed by He [52].We can obtainthe analytical solution of unsteady flow of the liquid through the nano-porous mediumby using this method.Assume that the approximate trial solution of equations (7) in the following form:

$$\boldsymbol{u(z) = a + b\, erf(mz)} \quad (17)$$

where a, b and m are constant. Using the boundary condition (14) the value of the constants are given by

$$a = 1 \text{ and } b = -1.$$

Now the eq. (17) becomes

$$u(z) = 1 - erf(mz) \quad (18)$$

Now to find the value of b, rewrite the eq. (18) as follows,

$$f(z) = \frac{d^2u(z)}{dz^2} + \frac{2z}{\sqrt{1-Au(z)}}\frac{du(z)}{dz} = 0 \quad (19)$$

From the eq. (19) we get

$$\frac{du(z)}{dz} = \frac{-2m}{\sqrt{\pi}}\exp(-m^2z^2) \quad \text{and} \quad \frac{d^2u(z)}{dz^2} = \frac{4zm^3}{\sqrt{\pi}}\exp(-m^2z^2) \quad (20)$$

Substituting this value in the eq. (19) we obtain

$$f(z) = \frac{4zm^3}{\sqrt{\pi}}\exp(-m^2z^2) - \frac{4zm}{\sqrt{\pi}}\exp(-m^2z^2) = 0 \quad (21)$$

Now, substituting $z = 1$,we get

$$f(z=1) = m^2 - \frac{1}{\sqrt{1-A}} = 0 \quad (22)$$

$$m = \frac{1}{(1-A)^{1/4}} \quad (23)$$

Now note that eq. (23) becomes

$$u_0(z) = 1 - erf\left(\frac{z}{(1-A)^{1/4}}\right) = erfc\left(\frac{z}{(1-A)^{1/4}}\right) \quad (24)$$

We can choose the function which satisfies the boundary condition $u_0(0) = 0, u_0(\infty) = 0$ for the differential equation $\frac{d^2u(z)}{dz^2} + \frac{2z}{\sqrt{1-Au(z=0)}}\frac{du(z)}{dz} = 0$ as follows:

$$u_1(z) = \frac{z}{2\pi}\exp(-2z^2) \quad (25)$$

Using eq.(24) and (25) we get

$$u(z) = u_0(z) + u_1(z) = erfc\left(\frac{z}{(1-A)^{1/4}}\right) + \frac{z}{2\pi}\exp(-2z^2) \quad (26)$$

Also from the equation we get at z=0,

$$\frac{du(z)}{dz} = \frac{-2m}{\sqrt{\pi}} + \frac{1}{2\pi} = \frac{-2}{\sqrt{\pi}}\left(\frac{1}{(1-A)^{1/4}}\right) + \frac{1}{2\pi} \tag{27}$$

These expressions are in closed analytical form which are very suitable for computation and indeed convey physical insight.

Previous existing analytical and numerical techniques for this problem are summarized in Table-1. Furthermore, Table 2 presents a variety of previously derived analytical expressions for *u(z)*. In Table-3, *u(z)* values derived using the expressions presented in the present work are compared with numerical and previous analytical results when *A*=0.5. Our closed form expressions are seen to generate reasonable results in good agreement with numerical analysis, although the error increases as z → 1. All of the previous analytical results which involve complicated multi-term series, were only moderately accurate. Also when the order of terms increases, the computations get more complicated, especially beyond the second-order term [12–29].

Table 1. Various analytical and numerical methods in the literature with numerical values of u'(0) when *A*=0.5.

S. No	Authors, Ref and Year	Methods (Analytical / Numerical)	Obtained value of u'(0)
1	Wazwaz [10], 2001	Pade approximation (Numerical)	−1.025529704
2	Parand et al. [13], 2009	Laguerre polynomials (Numerical)	−1.28213483
3	Taghavi et al. [14], 2009	Laguerre polynomials (Numerical)	−1.37317352
4	Noor and Mohyud-Din [15], 2009	Pade approximate (Analytical)	−1.025529704
5	Parand et al. [16], 2010	Chebyshev func (Numerical)	−1.38213483
6	Taghavi et al. [17], 2010	Lagrangian method (Numerical)	−1.37259357
7	Khan et al. [18], 2010	Laplace decomposition and Pade approximation(Analytical)	−1.373178096
8	Mohyud-Din et al. [19], 2010	He's polynomials and variational iteration (Analytical)	−1.025529704
9	Rezaei et al. [20], 2011	Sinc and Legendre function (Numerical)	−1.188692320
10	Rad et al. [21], 2011	Homotopy analysis (Analytical)	−1.188976708

11	Rad et al. [21], 2011	Hermite function (Numerical)	−1.183811 27860
12	Abbasbandy [22], 2012	Finite-difference and shooting method (Numerical)	−1.191790 771959
13	Kazem et al. [23], 2012	Radial Basis Function (Numerical)	−1.191498
14	Kazem et al. [23], 2012	Radial Basis Function (Numerical)	−1.191243
15	Parand and Nikarya [24], 2014	Bessel function (Numerical)	−1.191718 932
16	Upadhyay and Rai [25], 2014	Legendre wavelet (Numerical)	−1.199258 245
17	Wazwaz [26], 2014	Variational iteration Method (Analytical)	−1.025529 704
18	Iacono and Boyd [27], 2015	Chebyshev functions and Pade approximate (Numerical)	−1.192705 5
19	Parand and Hemami [28], 2015	Radial Basis Function (Numerical)	−1.191796
20	Kourosh Parand et al. [29], 2017	Rational Jacobi function (Numerical)	−1.191790 64
21	This work ,2021	New homotopy perturbation method,	- 1.182716897
22	This work,2021	Agbari-Ganji method	- 1.182736685

Table 2. Previous analytical results

s.no	Methods	Analytical expression of u(z)
1	ADM ((Wazwaz [10])	$u(z) = 1 + Bz - \frac{Bz^3}{3\sqrt{1-\alpha}} - \frac{\alpha B^2 z^4}{12(1-\alpha)^{3/2}} + \left(\frac{B}{10(1-\alpha)} - \frac{3\alpha^2 B^3}{80(1-\alpha)^{5/2}}\right) z^5 + \left(\frac{\alpha B^2}{15(1-\alpha)^2} - \frac{\alpha^3 B^4}{48(1-\alpha)^{7/2}}\right) z^6$, Where $B = -\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}}$
2	HPM (Jamal Amani Rad, Kourosh Parand [30])	$u(z) = 1 - erf(z) - \frac{(5\alpha^3 + 8\alpha + 6\alpha^2)z}{16\sqrt{\pi}e^{x^2}} + \frac{1}{64\pi^{3/2}e^{2x^2}} \begin{pmatrix} 2\alpha^2 e^{x^2}(5\alpha^3 + 6\alpha + 8)^2 \left(\frac{\alpha}{2} + \frac{3\alpha^2}{8} + \frac{5\alpha^3}{16}\right)^2 x^4 \pi^2 e^{x^2} + 16\left(2\alpha + 3\alpha^2 + \frac{15}{4}\alpha^3\right) erf(z) e^{x^2} \pi z \\ 16\left(2\alpha + 3\alpha^2 + \frac{15}{4}\alpha^3\right) erf(z) e^{2x^2} \sqrt{\pi} + 16\left(2\alpha + 3\alpha^2 + \frac{15}{4}\alpha^3\right)\sqrt{\pi} - 16\left(2\alpha + 3\alpha^2 + \frac{15}{4}\alpha^3\right)\sqrt{\pi}e^{2x^2} \end{pmatrix}$

3	Perturbation Series (Kidder[4])	$u(z)=u^{(0)}(z)-\frac{\alpha}{2\pi}\{u^{(0)}(z)[1+\sqrt{\pi}ze^{-z^2}]-e^{-2z^2}\}-\frac{\alpha^2u^{(1)}(z)}{\pi}+\left(\frac{1}{8\pi^{3/2}}\right)ze^{-3z^2}-\frac{\alpha^2u^{(0)}(z)}{2\pi}$ $-\left(\frac{1}{16\sqrt{\pi}}\right)z\alpha^2(5-2z^2)e^{-z^2}\left(u^{(0)}(z)\right)^2+\left(\frac{1}{4\pi}\right)(2-z^2)\alpha^2e^{-2z^2}u^{(0)}(z)$ $+\left(\frac{\alpha^2 3^{3/2}}{16\pi}\right)[erf(\sqrt{3}z)-erf(z)]$ Where $u^{(0)}(z)=1-erf(z)$,$u^{(1)}(z)=-\frac{1}{2\pi}\{u^{(0)}(z)[1+\sqrt{\pi}ze^{-z^2}]-e^{-2z^2}\}$
4	NHPM (This work)	$u(z)=erfc\left(\frac{z}{(1-A)^{1/4}}\right)$
5	AGM (This work)	$u(z)=erfc\left(\frac{z}{(1-A)^{1/4}}\right)+\frac{z}{2\pi}\exp(-2z^2)$

5. DISCUSSION

We developed analytical solutions for unsteady gas flow through porous media by utilizing both the homotopy perturbation method and the Akbari-Ganji method depending on all possible experimental values of parameter *A*. The derived analytical results were compared to numerical results generated by the highly accurate and reliable MATLAB's function pdex4. Tables 1-3 and Figures1&2 show that both methods produced a satisfactory analytical solution for the unsteady flow of gas through a porous medium. Furthermore our new analytical results exhibit less deviation (Average error percentage is 8% in NHPM and 2% in AGM) from the numerical results when compared to previously published results.

Table 3. Comparison of the values of *u (z)* with numerical and previous analytical results when *A*=0.5.

z	Numerical	AGM eq.(26)	Error AGM	NHPM eq.(16)	Error NHPM	KIDDER [4]	Error KIDDER	HPM [29]	Error HPM	PADE$_{[3,3]}$ [15]	Error PADE	GLP [16]	Error GLP
0.1	0.8802	0.8820	0.21	0.8825	0.26	0.8817	0.17	0.8881	0.90	0.8979	1.97	0.9004	2.24
0.2	0.7635	0.7660	0.33	0.7499	1.78	0.7663	0.37	0.7792	2.06	0.7985	4.39	0.8002	4.59
0.3	0.6528	0.6538	0.15	0.6387	2.16	0.6565	0.57	0.6760	3.55	0.7041	7.29	0.7081	7.81
0.4	0.5500	0.5474	0.48	0.5234	4.84	0.5544	0.79	0.5803	5.50	0.6165	10.79	0.6179	10.99
0.5	0.4565	0.4487	1.71	0.4351	4.69	0.4614	1.05	0.4933	8.07	0.5371	15.00	0.5339	14.50
0.6	0.3733	0.3594	3.72	0.3399	8.95	0.3783	1.32	0.4157	11.36	0.4666	19.99	0.4570	18.31
0.7	0.3006	0.2879	4.22	0.2730	9.18	0.3056	1.64	0.3475	15.59	0.4062	26.00	0.4074	26.22
0.8	0.2384	0.2273	4.64	0.2097	12.04	0.2431	1.95	0.2882	17.28	0.3561	33.05	0.3254	26.73
0.9	0.1860	0.1775	4.58	0.1542	17.10	0.1905	2.34	0.2372	21.60	0.3180	41.51	0.2707	31.29
1	0.1429	0.1296	9.29	0.1126	21.20	0.1588	10.00	0.1938	26.26	0.2900	50.73	0.2231	35.94
Error percentage			2.93		8.22		2.02		11.22		21.07		17.86

Table 4. Comparison of the values of $u'(0)$ with numerical and previous analytical results for various values of A.

A	$\frac{du(z)}{dz}$ $at\ z = 0$		
	Quasi-uniform grid	Rational Jacobi Function	NHPM/AGM eq. (16 & 26)
0.0	-1.128379137175471	---	-1.12837916709551
0.1	-1.139007180276811	-1.139007206178301	-1.15849571538969
0.2	-1.150475464757216	-1.150475486216286	-1.19311560505467
0.3	-1.162941442801447	-1.162941458295912	-1.23361740060298
0.4	-1.176615655957026	-1.176615666683335	-1.28208626184487
0.5	-1.191790644594857	-1.191790649719421	-1.34187653392775
0.6	-1.208894181745888	-1.208894174540914	-1.41886168642145
0.7	-1.228598484558365	-1.228598473695921	-1.52466610547822
0.8	-1.252083822445984	-1.252083790143917	-1.68732041202207
0.9	-1.281881374379111	-1.281881322203357	-2.006573439518899
1.0	-1.328230894324459	-----	-2.121693093492263

6. CONCLUSION

Unsteady isothermal gas equations play an important role in studying solid gases, and they are also a crucial element in the chemical process industries. The primary goal of this report is to provide a simple analytical and numerical algorithm for solving the unsteady gas flow equation in a semi-infinite porous media. The system of nonlinear differential equations was solved analytically and compared to the previous solution.

We have compared the value of $u'(0)$ with other numerical results because its value is significant in this problem.This method produces valid and accurate results for this type of problem.HPM and AGM are the powerful mathematical tool that can solve a high class of nonlinear differential systems, especially systems of nonlinear equations used in science and engineering.

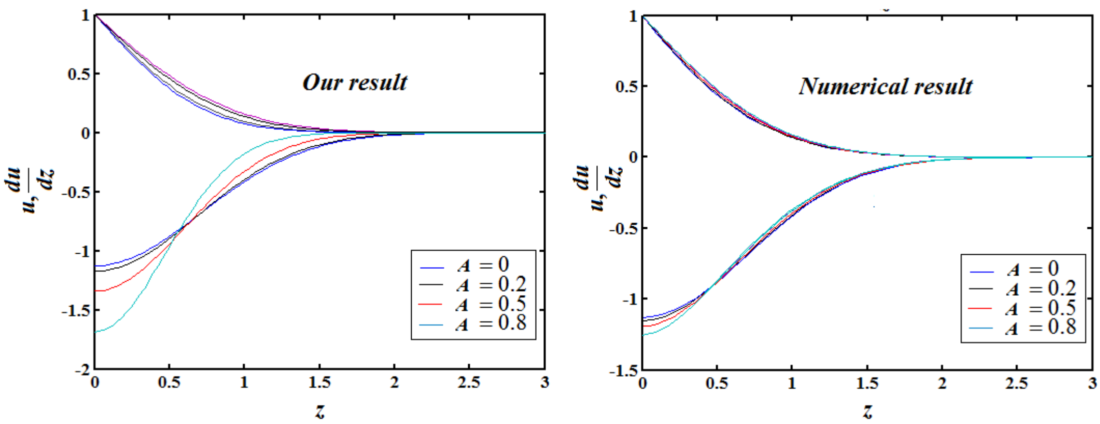

Figure1. Numerical and analytical solutions for the eq. (7) for increasing valuesof A

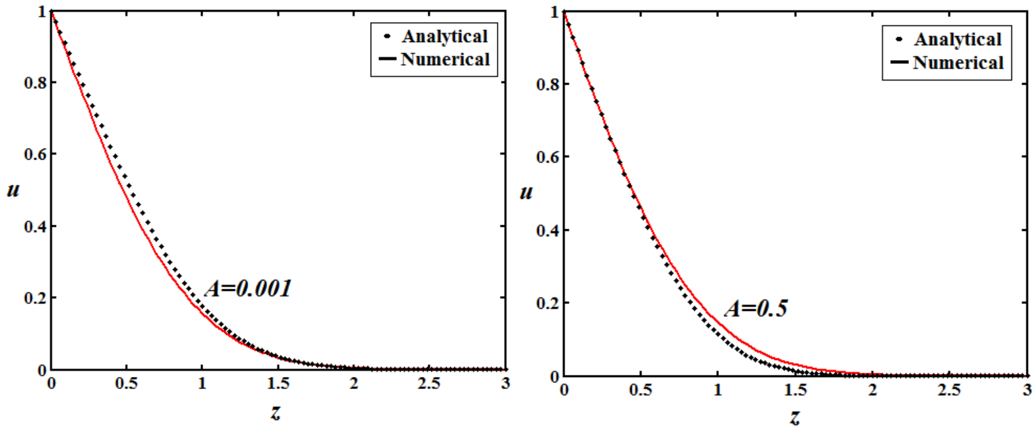

Figure 2. Comparison of concentration with numerical results for various values of parameter*A*.

APPENDIX A. NUMERICAL PROGRAM FOR THE SOLUTION OF NONLINEAR EQS. (7) - (9)

```
function sol=ex2
ex2init=bvpinit(linspace(0,1),[1 0]);
sol = bvp4c(@ex2ode,@ex2bc,ex2init)
end
functiondydx=ex2ode(x,y)
dydx=[y(2)
-y(2)*((2*x)/sqrt(1-(0.5)*y(1)))];
end
function res=ex2bc(ya,yb)
res=[ya(1)-1
yb(1)-0];
end
```

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